4050 Hw 4

*1. (Monotone Convergence Lemmas for measures). Let $m : \tilde{\mathfrak{M}} \to [0, +\infty]$ be a measure, where $\tilde{\mathfrak{M}}$ is an arbitrary σ-alg. Show that

(a) if $E_n \uparrow E$ (with each $E_n \in \tilde{\mathfrak{M}}$ $\forall n$), i.e. $E = \bigcup_{n \in \mathbb{N}} E_n$ & $E_n \subseteq E_{n+1}$ $\forall n$ then $m(E_n) \uparrow m(E)$.

(b) if $E_n \downarrow E$ (with - - - - - - - - - $E = \bigcap_{n \in \mathbb{N}} E_n$ & $E_n \supseteq E_{n+1}$ $\forall n$ then $m(E_n) \downarrow m(E)$ provided that $m(E_{n_0}) < +\infty$ for some n_0 (Hint: say $m(E_1) < +\infty$; apply (a) to $(E_1 \setminus E_n)$). Provide a counter-example if the added condition is dropped.

2. Let $\varphi := \sum_{i=1}^{n} a_i \chi_{E_i}$ (each $a_i \in \mathbb{R}$ & $E_i \in \mathfrak{M}$), be a "simple function"

(a) Show by MI that range(φ) is a finite set (and so one can list all its non-zero values $b_1, \cdots, b_N$ for some N unless φ is the zero-function); show that

$$\varphi = \sum_{j=1}^{N} b_j \chi_{\varphi^{-1}(b_j)} \quad \text{where } \varphi^{-1}(b_j) := \{x : \varphi(x) = b_j\}$$

and

$$\varphi = \sum_{j=0}^{N} b_j \chi_{\varphi^{-1}(b_j)} \quad \text{where } b_0 := 0 .$$

(both representations can be referred as $\begin{cases}\text{canonical}\\ \text{normal}\\ \text{standary}\end{cases}$ represention of φ)

(b) Define $\int \varphi := \sum_{j=1}^{N} b_j m(\varphi^{-1}(b_j)) = \sum_{j=0}^{N} b_j m(\varphi^{-1}(b_j))$ with the convention that $0 \cdot \infty = 0$) whenever $\varphi \in \mathcal{S}_0$ or $\varphi \in \mathcal{S}^+$ (meaning that the simple function φ is supported by a set of finite measure or $\varphi \geq 0$. Show that

2 (Continued) if $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$ with each $a_i \in \mathbb{R}$ and $m(E_i) < +\infty$ then $\int \varphi = \sum_{i=1}^{n} a_i \, m(E_i)$ provided that E_i's pairwise disjoint (the added condition can be dropped; see Q3 below).

Hint. Assume $a_i \neq 0 \; \forall i$. Then $\bigcup_{a_i = b_j} E_i = \varphi^{-1}(b_j) \; \forall j = 1, \cdots, N$

3. Let $\varphi, \psi \in \mathcal{S}_0$ with their canonical presentations $\varphi = \sum_{j=0}^{N} b_j \chi_{\varphi^{-1}(b_j)}$ and $\psi = \sum_{k=1}^{M} c_k \chi_{\psi^{-1}(c_k)}$. Show that $\int(\varphi + \psi) = \int \varphi + \int \psi$. Hint. Let $B_j := \varphi^{-1}(b_j)$ & $C_k := \psi^{-1}(c_k)$

Then $\varphi = \sum_j b_j \chi_{B_j} = \sum_j b_j \left(\sum_k \chi_{B_j \cap C_k} \right) = \sum_j \sum_k b_j \chi_{B_j \cap C_k}$

$\psi = \sum \sum c_k \chi_{C_k \cap B_j}$ and $\sum_j \sum_k (b_j + c_k) \chi_{B_j \cap C_k} = \varphi + \psi$

Now apply Q 2.

* 4. (3rd: P.70, Q21)

a. Let D and E be measurable sets and f a function with domain $D \cup E$. Show that f is measurable if and only if its restrictions to D and E are measurable.

b. Let f be a function with measurable domain D. Show that f is measurable iff the function g defined by $g(x) = f(x)$ for $x \in D$ and $g(x) = 0$ for $x \notin D$ is measurable.

* 5. (3rd: P.71, Q22)

a. Let f be an extended real-valued function with measurable domain D, and let $D_1 = \{x : f(x) = \infty\}$, $D_2 = \{x : f(x) = -\infty\}$. Then f is measurable if and only if D_1 and D_2 are measurable and the restriction of f to $D \setminus (D_1 \cup D_2)$ is measurable.

b. Prove that the product of two measurable extended real-valued functions is measurable.

c. If f and g are measurable extended real-valued functions and α a fixed number, then $f + g$ is measurable if we define $f + g$ to be α whenever it is of the form $\infty - \infty$ or $-\infty + \infty$.

d. Let f and g be measurable extended real-valued functions that are finite almost everywhere. Then $f + g$ is measurable no matter how it is defined at points where it has the form $\infty - \infty$.

*6. Let $\varphi \in \mathcal{S}_0$ and $E \in \mathcal{M}$. Define $\int_E \varphi := \int \varphi \cdot \chi_E$. Show that $\varphi \mapsto \int_E \varphi$ is linear:

$$\int_E (\alpha \varphi + \beta \psi) = \alpha \int_E \varphi + \beta \int_E \psi \quad \forall \varphi, \psi \in \mathcal{S}_0 \text{ and } \alpha, \beta \in \mathbb{R}.$$

*7. Let $\varphi \in \mathcal{S}_0$. Show that $A \mapsto \int_A \varphi$ is a ~~(new)~~ "signed" measure on $\mathcal{M}$. Hint: Do special case first with $\varphi = \chi_E$.